**Homework 10**

**P20.2.5** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.5, where *g*(*t*) = sin(*πt*/2), 0 ≤ *t* ≤ 2, and *g*(*t*) = 0 elsewhere. Determine *f*(*t*)\**g*(*t*) for all *t*. Verify the result by direct integration.



**Solution:** *Method 1* – *Graphical Solution*: When *f*(*t*) is folded



around the vertical axis and shifted by *t*, there are

four ranges of *t* to consider:

i) 0 ≤ *t* ≤ 1: *y*(*t*) = = .

ii) 1 ≤ *t* ≤ 2: *y*(*t*) = =



 .



iii) 2 ≤ *t* ≤ 3: *y*(*t*) =  ==



.

iv) *t* ≥ 3: there is no overlap between the

functions *y*(*t*) = 0.

*y*(*t*) will be as shown.

*Method 2* – *Analytical*: The square pulse is expressed as *u*(*t*) – *u*(*t* – 1), and the half-period as . Then *y*(*t*) is: = .

i) 0 ≤ *t* ≤ 1: Using Equation 20.4.1:  , as previously.

ii) 1 ≤ *t* ≤ 2: Using Equation 20.4.3, with *b* = 0 and *a* = 1,  . Adding this to the result of (i) gives , as previously.

iii) 2 ≤ *t* ≤ 3: Using Equation 20.4.3, with *b* = 2 and *a* = 0, =  = . Adding this to the result of (ii) gives , as previously.

iv) *t* ≥ 3: Using Equation 20.4.3, with *b* = 2 and *a* = 1,    Adding this to the result of (iii) gives 0, as previously.

*Method 3* – *Differentiating one function and integrating the other*: Differentiating the square pulse gives: *δ*(*t*) – *δ*(*t* – 1). The integral of the half period is as shown, and can be expressed as:  . Then *y*(*t*) is: .



i) 0 ≤ *t* ≤ 1: Using Equation 20.4.6: , as previously.

ii) 1 ≤ *t* ≤ 2: Using Equation 20.4.9, with *b* = 0 and *a* = 1: . Adding this to the result of (i) gives , as previously.

iii) 2 ≤ *t* ≤ 3: Using Equation 20.4.9, with *b* = 2 and *a* = 0: . Adding this to the result of (i) gives , as previously.

iv) *t* ≥ 3: Using Equation 20.4.9, with *b* = 2 and *a* = 1:  . Adding this to the result of (i) gives 0, as previously.

**P20.2.7** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.7. Evaluate *f*(*t*)\**g*(*t*) for all *t*. Verify the result using multiplication of polynomials.



**Solution:** When *f*(*t*) is folded around the vertical axis, it has



to be moved by 3 s

to the right before there is overlap with *g*(*t*). The first integration interval

is from *t* = 3 s to

*t* = 4 s. At *t* = 4 s, the area under the product is 3×2 = 6, and y(*t*) increases linearly



from 0 at *t* = 0 to 6 at *t* = 4 s. At *t* = 5 s, the net area is 2×3 – 2 ×1 = 4. At *t* = 6 s, the net area is -2 ×1 + 2×2 = 2. At *t* = 7 s, the net area is 2×2 = 4. At *t* = 8, there is no overlap, and *y*(*t*) = 0. *y*(*t*) is illustrated in the figure.

To apply multiplication of polynomials, the initial delays can be ignored, the polynomials multiplied together, and the resulting break points shifted by 3. Thus, *F*(*t*) = 2*t* + 2, and *G*(*t*) = 3*t*2 – *t* + 2; the product is 6*t*3 + 4*t*2 + 2*t* + 4. The break points of this polynomial are: (6, 1), (4, 2), (2, 3), (4, 4). Adding 3 to the time values gives the same result as before.

**P20.2.8** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.8. Evaluate *f*(*t*)\**g*(*t*) for all *t*. Verify the result using multiplication of polynomials.



**Solution:** After folding one of the functions around the vertical axis, and shifting by t, the values at the breakpoints are obtained for the various time intrvals/



For 0 ≤ *t* ≤ 1:

At *t* = 1, *y* = 9.

For 1 ≤ *t* ≤ 2:



At *t* = 2, *y* = 3×2 – 3×2 = 0.

For 2 ≤ *t* ≤ 3:



At *t* = 3, *y* = 1×3 – 2×2 + 3×1 = 2.

For 0 ≤ *t* – 3 ≤ 1, or 3 ≤ *t* ≤ 4:



At *t* = 4, *y* = -1×2 + 2×1 = 0.



For 1 ≤ *t* – 3 ≤ 2, or 4 ≤ *t* ≤ 5:

*A*t *t* = 5, *y* = 1× = 1.

For 2 ≤ *t* – 3 ≤ 3, or 5 ≤ *t* ≤ 6:

*A*t *t* = 6, *y* = 0.



The breakpoints are as shown. To use Matlab, enter f = [3,2,1] and

g = [3, -2, 1] corresponding to the levels of *f*(*t*) and *g*(*t*). Then enter: conv(f,g). Matlab returns: 9, 0, 2, 0, 1, corresponding to the breakpoints.

**P20.2.9** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.9. Evaluate *f*(*t*)\**g*(*t*) for all *t*. Verify the result using multiplication of polynomials.



Solution:

**Solution:** For 0 ≤ *t* ≤ 1:

*y* = *t*; at *t* = 1, *y* = 1.



For 1 ≤ *t* ≤ 2:



*y* = 2(*t* – 1) + 1(2 – *t*) – 2(*t* – 1) = 2 – *t*; at

*t* = 2, *y* = 0.



For 2 ≤ *t* ≤ 3:

*y* = 3(*t* – 2) + 2(3 – *t*) – 4(*t* – 2) – 2(3 – *t*) +

3(*t* – 2) = 2(*t* – 2) = 2*t* – 4; at *t* = 3, *y* = 2.

For 0 ≤ *t* – 3 ≤ 1, or 3 ≤ *t* ≤ 4:



*y* = 3(4 – *t*) – 6(*t* – 3) – 4(4 – *t*) + 6(*t* – 3) +

3(4 – *t*) = 8 – 2*t*; at *t* = 4, *y* = 0.



For 1 ≤ *t* – 3 ≤ 2, or 4 ≤ *t* ≤ 5:

*y* = -6(5 – *t*) + 9(*t* – 4) + 6(5 – *t*) = 9*t* – 36; at *t* = 5, *y* = 9.



For 2 ≤ *t* – 3 ≤ 3, or 5 ≤ *t* ≤ 6:

*y* = 9(6 – *t*); at *t* = 6, *y* = 0.

In terms of polynomials, *F*(*t*) = *t*2 + 2*t* + 3, and *G*(*t*) = *t*2 – 2*t* + 3. The product is *y*(*t*) = *t*4 + 2*t*2 + 9. The breakpoints are as shown. To use Matlab, enter f = [1,2,3] and g = (1, -2, 3] corresponding to the levels of *f*(*t*) and *g*(*t*). Then enter: conv(f,g). Matlab returns: 1, 0, 2, 0, 9, corresponding to the breakpoints.



**P20.2.10** Given *f*(*t*) and *g*(*t*) as in Figure P20.2.10, where *g*(*t*) = sin*π*(t - 1), 1 ≤ *t* ≤ 2, and *g*(*t*) = 0 elsewhere. Determine *f*(*t*)\**g*(*t*) for all *t*. Verify by convolving with step functions.



**Solution:** sin*π*(*t* – 1) = -sin*πt*. When *g*(*λ*) is folded with respect to the vertical axis and shifted to the right by *t*, *y*(*t*) = 0 for *t* ≤ 1.

For 1 ≤ *t* ≤ 2, the graphical construction will be as shown, and **.



For 2 ≤ *t* ≤ 3, *y*(*t*) remains constant and equals **.

For 3 ≤ *t* ≤ 4, ** **.



*y*(*t*) = 0 for *t* ≥ 4 and will be as shown.

Alternatively, *y*(*t*) can be derived as the convolution of sinusoidal and step functions. Thus, *f*(*t*) = *u*(*t*) – *u*(*t* – 2), and *g*(*t*) = ** + *.* It follows that: *y*(*t*) = ** + *.* These terms will be convolved using Equation 20.4.2. Thus, with *b* = 1 and *a* = 0,



**= **

**. It follows that *y*(*t*) = 0 for 0 ≤ *t* ≤ 1 and **, as before, for 1 ≤ *t* ≤ 2.

For 2 ≤ *t* ≤ 3, * * **. Adding this to the previous result gives **, as before.

For 3 ≤ *t* ≤ 4, -**= ** **. Adding this to the previous result gives **, as before.

For *t* ≥ 4, -** = ** **. Adding this to the previous result gives *y*(*t*) = 0, as before.



**P20.2.11** Given *f*(*t*) and *g*(*t*) in Figure P20.2.11. Determine *h*(*t*)\**x*(*t*).

**Solution:** For 0 ≤ *t* ≤ 1:

*y* = = = = .



For 1 ≤ *t* ≤ 2:



*y* = + + = + + = + + =. It is seen that *y*(1) = 5/6 and *y*(2) = 4/3.

For 0 ≤ *t* – 2 ≤ 1, or 2 ≤ *t* ≤ 3:



*y* = +

+ = + + = + + = ; it is seen that *y*(2) = 4/3 and *y*(3) = 5/6.



For 1 ≤ *t* – 2 ≤ 2, or 3 ≤ *t* ≤ 4:

*y* = = = = ; it is seen that *y*(3) = 5/6 and *y*(4) = 0. Moreover, *y*(*t*) = 0 for *t* ≥ 4.

**P20.2.13** Given *f*(*t*) and *g*(*t*) in Figure P20.2.13. (a) Evaluate *f*(*t*)\**g*(*t*); (b) If the impulse response of a circuit is , determine the response of the circuit to *f*(*t*).



**Solution:** (a) When *f*(*t*) is folded around the vertical axis, it coincides with *g*(*t*). It must be shifted in the negative direction by 3 units to start the integration at *t* = -3 units. As *f*(-*λ*) is shifted to the right, the convolution integral increases linearly, reaching a maximum at *t* = 0, when the two functions coincide, then decreasing linearly to zero after 3 units.



b) The response to *u*(*t*) is . The response to *f*(*t*) is the sum of the responses to two step functions: .



**P20.2.14** Given *f*(*t*) and *g*(*t*) in Figure P20.2.14. Evaluate *f*(*t*)\**g*(*t*) for all *t*.

**Solution:** When f(t) is folded around the vertical axis, it remains in position with no overlap between the functions, and *y*(*t*) = 0. At *t* = 1 s, the area of overlap is -1. At *t* = 2 s, the net area is zero. At *t* = 3 s, the area of overlap is 1, and at *t* = 4, there is no longer any overlap and *y*(*t*) = 0. *y*(*t*) is as shown. *y*(*t*) is given by:



*y*(*t*) = -*t*, 0 ≤ *t* ≤ 1; *y*(*t*) = (*t –* 2*)*, 1 ≤ *t* ≤ 3; *y*(*t*) = -(*t –* 4*)*, 3 ≤ *t* ≤ 4; *y*(*t*) = 0, *t* ≥ 4.

